Computed Tomography

Volume 27

PROCEEDINGS OF SYMPOSIA IN APPLIED MATHEMATICS

AMERICAN MATHEMATICAL SOCIETY

Proceedings of Symposia in Applied Mathematics
Volume 27, 1982

COMPUTED TOMOGRAPHY: SOME HISTORY AND RECENT DEVELOPMENTS

A.M. Cormack

My early papers [7,8] on what is now called computed tomography were entitled "Representation of a Function by its Line Integrals, with some Radiological Applications", and in one of them I wrote "One would think that this problem would be a standard part of the nineteenth century mathematical repertoire, but the author has found no reference to it in standard works". I have still found no nineteenth century reference to the problem, but in the years since 1971 I have come across a number of early discoveries of Radon's problem in the physical sciences, which I would like to share with you since they are not generally known. For more mathematical references, see John [21] and Helgason [19]. In addition, I will sketch my connection with the problem and give some results which I have recently obtained.

The first person I know of who tackled Radon's problem was the great Dutch physicist H.A. Lorentz. He found a solution to the three dimensional problem where a function is to be recovered from its integrals over planes. If $\hat{f}(p,\vec{n})$ is the integral of f over a plane perpendicular to the vector $p\vec{n}$ from the origin, and a distance p from the origin, then f at the origin is given by

$$f(0) = -\frac{1}{8\pi^2}\int\left(\frac{\partial^2 \hat{f}(p,\vec{n})}{\partial p^2}\right)_0 d\omega_n$$

where $d\omega_n$ is an element of solid angle in the direction of $\vec{n}$. Since the origin may be arbitrarily chosen, the result holds for any point. We have no idea why Lorentz thought of the problem, or what his method of proof was, and the only reason we know of his work is that the above result was attributed to him by Bockwinkel [4] who used the result in a long paper on the propagation of light in biaxial crystals.

Following Lorentz came Radon's paper, but since all of you have copies of it, I shall not comment on it.

1980 Mathematics Subject Classification 01A60, 44A15, 45A05

0160-7634/82/0000-0876/$03.00

The problem appears again in Holland in 1925 and again involves the physicists Paul Ehrenfest and George Uhlenbeck (who is still alive in this country). In Uhlenbeck's paper he says that Ehrenfest drew his attention to the result of Lorentz and suggested that he generalize it to n-dimensions, which he proceeded to do using Fourier techniques [33]. No reason for solving the problem was given, and my guess is that it was just a cute problem crying out for a solution. I am indebted to Alberto Grünbaum for drawing my attention to the Bockwinkel and Uhlenback papers.

1936 was a vintage year for it contained two independent discoveries of Radon's problem. One was in Stockholm where Cramér and Wold [15] asked the following question: given all marginal distributions of a probability distribution, can one infer the distribution itself? Marginal distributions are what people in computed tomography call projections or views, so the question is just Radon's problem, and in solving it, Cramér and Wold again used the Fourier integral approach. The second was in Leningrad where the well-known Armenian astronomer, V.A. Ambartsumian, posed the following question which he attributes to Eddington [3]. If one looks in a particular direction in space, one sees many stars and these are moving about relative to one another and to the sun. Astronomers would like to know their distribution of velocities but, except for a few close stars, one can only measure their radial velocities, which are deduced from the Doppler shifts of their spectra. The problem then is to deduce the actual distribution of velocities in three dimensions in space from the distributions of radial velocities in various directions. (It is, of course, necessary to assume the velocity distribution to be invariant under spatial translations.) This is just Radon's problem in a three dimensional velocity space rather than ordinary space, and Ambartsumian gave the solution in two and three dimensions [1] in the same form as Radon. Furthermore, he took groups of stars of three different spectral types, with four to five hundred stars in each group, and he used his theoretical results to deduce their actual velocity distributions from the distributions of their radial velocities [2]. One of his results is shown in Fig. 1 and it is in fact the velocity distribution projected onto the galactic plane. This is the first numerical inversion of the Radon transform and it gives the lie to the often made statement that computed tomography would be impossible without computers. Details for the calculation are given in Ambartsumian's paper, and they suggest that even in 1936 computed tomography might have been able to make significant contributions to, say, the diagnosis of tumors in the head.

The problems of finding abnormalities in the soft tissues from X-rays, even with the aid of computed tomography, are far greater than most people realize so it seems to me quite possible that Ambartsumian's numerical methods

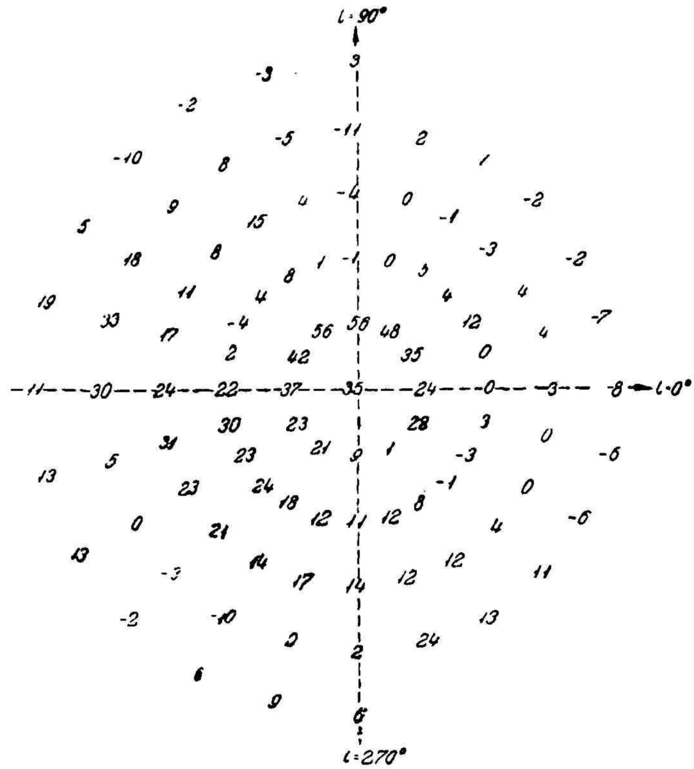

Figure 1
Velocity distribution of stars projected onto the galactic plane.

might have made significant contributions to that part of medicine had they been applied in 1936. I am indebted to Professor Ambartsumian for drawing my attention to his work and for telling me that he was informed of Radon's results two years after he published his work.

In 1947 there is a fascinating abstract by Szarski and Wazewski [32] which begins: "En disposant des fotographies roentgeniques de la tête humaine de tous les côtés, peut-on déterminer la structure intérieure du crâne, voici un problème proposé par M. Stanislaw Majerek, médicin de Cracovie qui, sous un forme élémentaire, a indiqué un méthode de solution qui nécessite une justification mathématique". They then go on to describe Radon's problem and formulate it in terms of a set of "fonctions cylindrique" and state that the problem consists of finding whether this set of functions tends to a solution of Radon's problem. I have been unable to find any further references to this work.

If one looks at the sun, as Bracewell was doing around 1956, with a long radio antenna with a parabolic section perpendicular to its length the signal one gets is an average over a thin strip of the sun of its radio emission at some wavelength. On moving the antenna one gets averages over different strips,

and once again we have Radon's problem, which Bracewell proceeded to solve using Fourier and other methods [5].

As I have recounted elsewhere [10] in 1956 I came across the problem in radiology where one has to determine the X-ray absorption coefficient from point to point in a disc from attentuation measurements on X-ray beams traversing the disc; Radon's problem again! Since the treatment I gave differs from previous treatments, and since it shows some interesting features which appear later, I will outline it.

First consider a circularly symmetrical function $f(r)$. Then if $\hat{f}(p)$ is the line integral along a line which is a perpendicular distance p from the origin, it is easy to show that

$$\hat{f}(p) = 2\int_p^\infty \frac{f(r)rdr}{\sqrt{r^2-p^2}}. \tag{1}$$

This is just Abel's equation with the well-known solution

$$f(r) = -\frac{1}{\pi}\int_r^\infty \frac{\hat{f}'(p)dp}{\sqrt{p^2-r^2}} \tag{2}$$

which is quite amenable to numerical analysis and which I used in an experiment [7]. The fact that the integral in Eq. (2) extends from r to ∞ is an explicit expression of the hole theorem: to find $f(r)$ one needs only $\hat{f}(p)$ for $p \geq r$. This remains true for non-circularly symmetrical functions as can be seen from the following. Let the line L along which $\hat{f}$ is calculated be specified by (p,ϕ) as in Fig. 2.

Reconstruction of Densities from their Projections

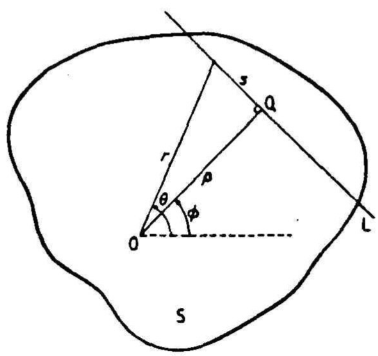

Fig. 2. Coordinate system defining the line L in a slice of the object. The line L is the limit of the cylinder C of fig. 1.

Figure 2.

Coordinates for the Radon transform in R^2

Then if one writes

$$f(r,\theta) = \sum_\ell f_\ell(r)e^{i\ell\theta}, \qquad \hat{f}(p,\phi) = \sum_\ell \hat{f}_\ell(p)e^{i\ell\phi},$$

one can show that

$$\hat{f}_\ell(p) = 2\int_p^\infty \frac{f_\ell(r)T_\ell(p/r)rdr}{\sqrt{r^2-p^2}}, \tag{3}$$

which has the solution

$$f_\ell(r) = -\frac{1}{\pi}\int_r^\infty \frac{\hat{f}'_\ell(p)T_\ell(p/r)dp}{\sqrt{r^2-p^2}}, \tag{4}$$

where $T_\ell(x)$ is the Tschebicheff polynomial of the first kind which is orthogonal in (-1,1) [8]. In Eq. (4), $T_\ell(x)$ appears for $x \geq 1$, and its leading term is const $\times\ x^\ell$ hence if $\hat{f}'_\ell(p)$ is obtained from noisy data, the noise is propagated badly from the exterior in as one goes to smaller values of r. In a recent application of this method in fusion physics [30] it is shown that Eq. (4) gives bad results for $\ell \gtrsim 10$. Hence Eq. (4) is not good for numerical solutions in medicine where one would require ℓ to be of the order of hundreds. I therefore resorted to radial expansions of f_ℓ and $\hat{f}_\ell$ which take the following form. If

$$f_\ell(r) = r^\ell(1-r^2)^{\lambda-1/2}\ G_k(\ell+\lambda,\ \ell+1,\ r^2), \tag{5a}$$

then

$$\hat{f}_\ell(p) = \text{const} \times\ (1-p^2)^{\lambda-1/2}\ C^\lambda_{\ell+2k}(p), \tag{5b}$$

where the G_k are Jacobi polynomials in (0,1) and the C^λ_ℓ are Gegenbauer polynomials. This expansion, for general λ, was recently given by Louis [22] and I used only the case $\lambda = 1$ which has been discussed by Marr [24], Ein-Gal [17] and others. Application of these formulae in an experiment to find f from $\hat{f}$ showed that computed tomography would work well for real data, but today the most commonly used numerical methods are based on the convolution algorithm of Ramachandran and Laksminarayanan [29].

When Radon's problem is extended to R^n Ludwig [23] and Deans [16] obtained formulae which contain Eqs. (3) and (4) as special cases. For more general spaces, a more general form of the hole theorem is immediately obtainable from a theorem (2.1) of Helgason [18].

Since the middle sixties there has been an explosion in applications of Radon's problem to many problems in science and engineering some of which are given in Marr [25] and more in a hard-to-obtain bibliography by Richard Gordon. Since the advent of commercial computed tomography scanners [20] there has been a greater explosion in the medical field, and reviews of Brooks and DiChiro [6] and Shepp and Kruskal [31] are useful, though both are becoming dated.

Radon's problem in R^2 gives rise to a nice tiling problem which has not been used and which may not even be useful. Consider two pairs of points, S_1,S_2 and S'_1,S'_2 which define a beam of X-rays traversing a sample in the following way: only lines which intersect S_1S_2 and $S'_1S'_2$ contributed to a measurement

of some average value of f with this beam. The lines defined by S_1S_2 and $S_1'S'_2$ are represented by a region of the Radon transform bounded by four circles, namely, those with radii OS_1, OS_2, OS_1', OS_2' where O is the origin of the transform. The question then is how the Radon transform of a finite object can be covered exactly by a set of such beams which intersect it. The solution is easy for a model of positron-annihilation (PET) scanning when the detectors can be considered to be arcs of the circle defining the object [11]. For exactly parallel beams (i.e. S_1S_2 and $S_1'S_2'$ forming the corners of a fixed rectangle) there is no exact solution, but a mix of parallel and almost-parallel beams does provide a solution [9]. One wonders whether there are other solutions, and whether these results can be extended to R^n.

When I had obtained a solution of Radon's problem in the form given in Eq. (4), I asked myself whether there were other curves specified by parameters (p,ϕ) for which Radon's problem could be solved. I immediately tried circles through the origin with diameter p and orientation of the diameter relative to the x-axis given by ϕ, and obtained results remarkably similar to Equations (3) and (4), namely

$$\hat{f}_\ell(p) = 2\int_0^p \frac{f_\ell(r)T_\ell(r/p)dr}{\sqrt{p^2-r^2}}, \tag{6}$$

which has the solution

$$f_\ell(r) = \frac{1}{\pi}\frac{d}{dr}\int_0^r \frac{\hat{f}_\ell(p)T_\ell(r/p)dp}{\sqrt{r^2-p^2}}. \tag{7}$$

Quinto and I recently generalized these to R^n and applied the results to solutions of Darboux' equation [14]. The results will not be given here, but the kernels are more complicated than those in Eqs. (6) and (7), involving Gegenbauer polynomials which appear here as in [23] and [16] as a result of using the Funk-Hecke theorem for spherical harmonic expansions.

As an older physicist, I was aware that in electrostatics straight lines and planes are related to circles and spheres by inversion $((r,\theta) \to (1/r,\theta))$ and this connection intrigued me for a long time. Recently I came across two families of curves which are related by inversion, and for which Radon's problem can be solved, which include the straight lines and circles. These are, in the previous notation,

$$\begin{aligned} r^\alpha \cos\{\alpha(\theta-\phi)\} &= p^\alpha,\ \alpha > 0 \\ p^\beta \cos\{\beta(\theta-\phi)\} &= r^\beta,\ \beta > 0. \end{aligned} \tag{8a}$$

For $\alpha = 1/2, 1, 2$ we have parabolae, straight lines and hyperbolae respectively, and for $\beta = 1/2, 1, 2$ we have cardioids, circles through the origin and one branched lemniscates of Bernoulli, respectively. Many of these curves are the orbits of particles in central fields or of charged particles in certain

magnetic fields. The details can be found in [12], but in general the results are very like those for the ordinary Radon transform: the inversion formulae are similar and the appropriate hole theorems make their appearance. More recently [13] certain other properties have been obtained. For general values of α and β certain inverse and direct powers of r ($r \neq 0$) yield null Radon transforms, as pointed out by Newman [26] and Louis [22] for the ordinary Radon transform. For $\alpha = 1$, $\beta = 1$, Quinto [28] has shown that these constitute the entire null space. For α or $\beta = 1/m$, $m = 1, 2, 3...$ a number of orthogonal expansions of f_ℓ and $\hat{f}_\ell$ are given. For α-curves in (0,1) the expansions are an extension of the results of Louis (Eqs. 5a,5b); in $(1,\infty)$ they are an extension of Perry's results [27]. In $(0,\infty)$ they follow the pattern given in [8]. For β-curves, the inversion property allows one to write down immediately the appropriate formulae for (0,1) and $(1,\infty)$ from the results for α-curves for $(1,\infty)$ and (0,1) respectively.

Working on Radon's problem has given me a great deal of pleasure and, as usual, each question answered has raised new questions. There are still plenty of these to be answered.

BIBLIOGRAPHY

1. Ambartsumian, V., M.N.R.A.S., 96 (1936), 172.
2. Ambartsumian, V., Publ. de L'Obs. Astron. Leningrad 7 (1936) 22.
3. Ambartsumian, V., Ann. Rev. Astron. Astro. 18 (1980) 1.
4. Bockwinkel, H.B.A., Verh. Kon. Akad. Wet. Wissen. Naturk. 14 (1906) 636.
5. Bracewell, R.N., Austr. J. Phys. 9 (1956) 198.
6. Brooks, A.R., and DiChiro, G., Phys. Med. Biol. 21 (1976) 689.
7. Cormack, A.M., J. Appl. Phys. 34 (1963) 2722.
8. Cormack, A.M., J. Appl. Phys. 35 (1964) 2908.
9. Cormack, A.M., Phys. Med. Biol. 23 (1978) 1141.
10. Cormack, A.M., Science 209 (1980) 1482.
11. Cormack, A.M., Phys. Med. Biol. 25 (1980) 543.
12. Cormack, A.M., Proc. Am. Math. Soc. 83 (1981) 325.
13. Cormack, A.M., to appear in Proc. Am. Math. Soc.
14. Cormack, A.M., and Quinto, E.T., Trans. Am. Math. Soc. 260 (1980) 575.
15. Cramér, H. and Wold, H., J. Lond. Math. Soc. 11 (1936) 290.
16. Deans, S., J. Math. Phys. 19 (1978) 2346.
17. Ein-Gal, M., Stanford U. Information Systems Laboratory, Technical Report 6851-1 (1974).
18. Helgason, S., Acta Math. 113 (1965) 153.
19. Helgason, S., The Radon Transform, Birkhauser, Boston (1980).
20. Hounsfield, G.N., Br. J. Radiol. 46 (1973) 1016.

21. John, F., Plane Waves and Spherical Means, Interscience, New York (1955).

22. Louis, A.K., SUNY, Buffalo, Medical Image Processing Report T52 (1981).

23. Ludwig, D., Comm. Pure Appl. Math. 69 (1966) 49.

24. Marr, R.B., J. Math. Anal. Appl. 45 (1974) 357.

25. Marr, R.B., Brookhaven National Laboratory, BNL 20425 (1974).

26. Newman, D.J., cited in reference 31.

27. Perry, R.M., J. Math. Anal. Appl. 59 (1977) 324.

28. Quinto, E.T., to appear in J. Math. Anal. Appl..

29. Ramachandran, G.N. and Laksminarayanan, A.V., Proc. Math. Acad. Sci. U.S.A., 68 (1971) 2236.

30. Sauthoff, N.R. and von Goeler, S., I.E.E.E. Trans. on Plasma Science 7 (1979) 141.

31. Shepp, L.A. and Kruskal, J.B., Am. Math. Monthly 85 (1978) 420.

32. Szarski, J., and Wazewski, T., Ann. Soc. Polon. de Math. 20 (1947) 389.

33. Uhlenbeck, G.E., Physica 5 (1925) 423.

DEPARTMENT OF PHYSICS
TUFTS UNIVERSITY
MEDFORD, MASSACHUSETTS 02155